VŠB – Technical University of Ostrava
Faculty of Electrical Engineering and Computer Science
Department of Applied Mathematics

Cellular automata and CML systems

Celulární automaty a CML systémy

2018

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Bachelor Thesis Assignment

Student:	**Václav Štula**
Study Programme:	B2647 Information and Communication Technology
Study Branch:	1103R031 Computational Mathematics
Title:	Cellular automata and CML systems Celulární automat a CML systémy
The thesis language:	English

Description:

The cellular automata are used in engineering applications. Connections between cellular automata and dynamical system on a lattice (Coupled Map Lattice systems), as a mathematical tool, were studied. The main aim of the thesis is to study elementary properties of CML systems and their conversion to cellular automata.

The tools for the thesis:
1. study the notion,
2. find out a suitable application,
3. construct an algorithm motivated by a suitable CML system,
4. conclusions.

References:

[1] Chazottes, Jean-René; Fernandez, Bastien, Dynamics of Coupled Map Lattices and of Related Spatially Extended Systems, Springer, 2005.

[2] E. Golès, Servet Martínez, Cellular automata, dynamical systems, and neural networks, Kluwer Academic Publishers, 1994.

[3] Kai Nagel and Michael Schreckenberg, A cellular automaton model for freeway traffic, J. Phys. I France 2 (1992) 2221-2229.

Extent and terms of a thesis are specified in directions for its elaboration that are opened to the public on the web sites of the faculty.

Supervisor: **doc. RNDr. Marek Lampart, Ph.D.**

Date of issue: 01.09.2017

Date of submission: 30.04.2018

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I hereby declare that this bachelor's thesis was written by myself. I have quoted all the references I have drawn upon.

Ostrava, 30th of april 2018 ....

I would like to thank my parents and my closest friends for their support as well as doc. RNDr. Marek Lampart, Ph.D. for his advice and patience.

Abstrakt

Hlavním cílem práce je studium vztahu celulárních automatů a diskrétních dynamických systémů na mřížce. Oba nástroje, jak celulární automat tak dynamický systém na mřížce, jsou zavedeny a jejich základní vlastnosti popsány. Vztah mezi celulárními automaty a dynamickými systémy na mřížce je podrobně popsán. Hlavním cílem práce je dále použití nástroje celulárního automatu jako matematického vizualizačního prostředku evoluce diskrétních dynamických systémů. Teorie celulárních automatů je použita na dynamické systémy na mřížce Lamplaceova typu a implementována v prostředí Java.

Klíčová slova: dynamické systémy na mřížce, celulární automaty, logistický systém zobrazení

Abstract

The main aim of this thesis is the study of cellular automata and discrete dynamical systems on a lattice. Both tools, cellular automata as well as dynamical systems on a lattice are introduced and elementary properties described. The relation between cellular automata and dynamical system on lattice is derived. The main goal of the thesis is also the use of the cellular automata as that mathematical tool of evolution visualization of discrete dynamical systems. The theory of cellular automata is applied to the discrete dynamical systems on a lattice Laplacian type and implemented in Java language.

Key Words: coupled map lattices, cellular automata, logistic family

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CML	–	Coupled map lattice
CA	–	Cellular automaton
Fix	–	Fixed point
Per	–	Periodic point

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1 Introduction

Cellular automaton (CA) is a discrete model used in many scientific fields like computer science, mathematics, physics, theoretical biology etc. CA contains grid of cells. Those grids could be n-dimensional but in general are two or three dimensional. Cells have finite number of states like for example 0 or 1. For each cell is defined a neighborhood, that is a set of cells which could be defined differently for each CA. Usually neighbor is cell with common edge in grid.

A coupled map lattice (CML) is a model for the time evolution of nonlinear systems. They are mainly used to model a suitable physical reality and to study the chaotic dynamics of spatially extended systems, which are extended in space or involve a lot of individual units. For CML is typical, that they are discrete in time with discrete state and continuous actions.

Very interesting is, that some CA patterns can be observed in nature, for example some seashells, like the ones in Conus or Cymbiola genus, are generated by cellular automata. Every cell secretes pigment according to activity of its neighbor cells and obeys a mathematical rule from nature. As shell was growing, it left a colored pattern. For example Conus textile is colored according to Wolfram's rule 30 (Coombes, 2009). See in Figure 1

As mentioned, CA could be used in different branches. One of the possible application are computer processors. They are CA's physical implementations, which process information. Elements in process are in regular grid of identical cells. Grid is usually rectangular or made out of tiles and most likely two or three dimensional (Muhtaroglu, 1996). Different shapes of cells are possible but not used. States of cells are determined only by interaction with neighbours. There is no possible way, that cell will communicate with distant cell.

Figure 1: Conus shell

Another application is Cryptography. Some of two dimensional CA are used as random number generator and for example Rule 30 was suggested for use in cryptography as a block cipher. The next generation is quite easy to find, but previous generation are very difficult to calculate, so we can use them as private and public keys in cryptography(Tomassini, Sipper, & Perrenoud, 2000).

CA were used also for error correction codes in paper by ((Chowdhury, Basu, Gupta, & Chaudhuri, 1994))

The main aim of the thesis is to use the tool of cellular automaton as visualization instrument to show the evolution of the dynamical systems. For this purpose detailed overview of the theory of cellular automata is deeply researched and applied to the coupled map lattice model of the Laplacian type. Implementation of the cellular automaton was performed in Java language and the output as a flip book is situated in bottom right corner (Figures 2 - 4,6 - 9, 12 - 14 and 16).

This thesis is organized as follows. In the Section 2 a general framework of cellular automaton is given, moreover elementary cellular automata are introduced and the theory of cellular automata are computed. Section 3 is devoted to the theory of coupled map lattices. The logistic family is introduced, as well as CML system of the Laplacian type, and the implementation of CML as CA is delineate, where evolution of CML system generated by CA is derived. Section 4 describes the implementation of CML as CA in Java and Section 5 is closing the thesis with concluding remarks.

2 Cellular automata

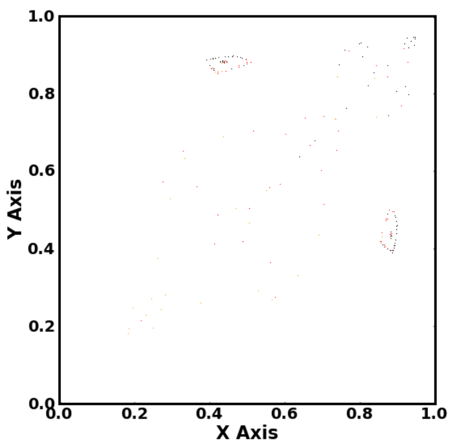
As mentioned in the previous section, cellular automaton can be interpreted as a computer model, which is very useful in many ways. The first important principle of CA is its spacial structure. Each square or cell has its location. This could represent for example a city with buildings and people in it. One cell could be a house. As in real world, each cell has neighbours. Neighborhood is not always the same for every cellular automaton (Pickover, 2009; Von Neumann, 1951; Von Neumann & Burks, 1996). Each cellular automaton could have defined neighborhood differently. For example von Neumann neighborhood consists of four neighbours, one on the bottom, one on the top, one on the left, and one on the right. Then we have for example Moore neighborhood, which consists of eight cells so also cells bordering with corners are considered as neighbors. By defining the neighborhood we decide, how big impact have interactions between cells(Goles & Martínez, 2013; Kier, Seybold, & Cheng, 2005). For example a disease in von Neumann neighborhood will spread not as fast as in Moore neighborhood. The second important principle is local interactions.

In modeling there are another two basic principles. First one is that cells have a state. This could represent for example an opinion of a person or condition of an organism. State is typically modeled as a number. Second important principle is that states can change. How can they change their state depends on both them and their neighbors. For example if people around you are infected by a disease, you will probably get infected too (Toffoli & Margolus, 1987; Schiff, 2011). In CA time is discrete, so time moves on with steps. In each step CA controls every cell, if they change its state.

Elementary CA are one-dimensional binary CA with neighborhood of size one. The rule to determine the next state of a cell depends only on the current state of the cell and its two neighbors. This is one of the simplest models. There are 8 possible combinations of cell and its neighbors. The rule specify every combination, so there is 256 elementary cellular automata. We cannot predict behavior of CA, we have to observe it. There are four possible outcomes of CA (Amigó, 2010):

- the configurations converge to a fixed point,
- time evolution yields a sequence of simple stable or periodic structures,
- the behavior is chaotic,
- time evolution yields localized structures that move around and interact with each other in very complicated ways.

Figure 2: The eleventh iteration of the CML model (13) as CA for $\mu = 3.8$ and $\epsilon = 0.06$



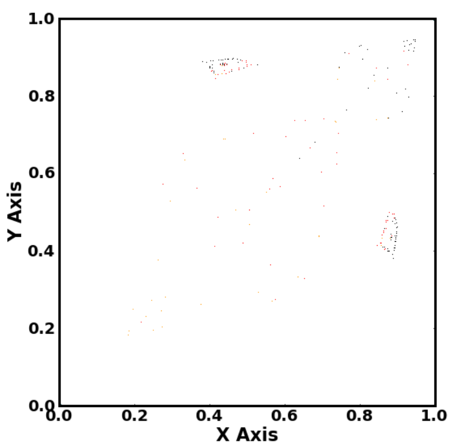
The most famous cellular automaton is Conway's Game of Life. With this CA came up British mathematician John Horton Conway in 1970. Game of life has two-dimensional grid of cells, where each cell have one of two possible states, alive or dead (one or zero, populated or unpopulated,...). Each cell has eight neighbours, this could be also called Moore neighborhood. Rules are quite simple and are valuated for each cell in every step:

- a cell with two or less neighbours dies due to underpopulation,
- a cell with two or three neighbours continues to live,
- a cell with more than three neighbours dies due to overpopulation,
- a dead cell with exactly three neighbours become alive, as if by reproduction.

Firstly we apply these rules on initial pattern and then on each generation. Game usually ends with extinction. This game rase interest in CA and in 1980s Stephen Wolfram begin with systematic study of elementary CA (Gardner, 1970).

A lot of people have asked a question, whether the universe is a cellular automaton. Andrew Ilachinski (Ilachinski, 2001) argues that the importance of this question may be better appreciated with a simple observation, which can be stated as follows. Consider the evolution of rule 110: if it were some kind of "alien physics", what would be a reasonable description of the observed patterns? If an observer did not know how the images were generated, that observer might end up conjecturing about the movement of some particle-like objects. And Ilachinski is not the only one. For example physicist James Crutchfield (Crutchfield, 1994) has constructed a mathematical theory based on this idea. Question is whether our world, which is mostly described, with our current knowledge, by physics, could be a CA at its most basic level with the gaps in information (Kemeny, 1955; Chapman, 2002; Mitchell, 2002).

Figure 3: The tenth iteration of the CML model (13) as CA for $\mu = 3.8$ and $\epsilon = 0.09$



Those discussions and hypothesis led to speculations on how we can make sense of our world in a discrete framework. Marvin Misnki studied particle interactions using a four-dimensional CA lattice, Konrad Zuse came up with irregularly organized lattice and asked question of the information content of particles. Edvard Fredkin discovered the finite nature hypothesis. This hypothesis says, that "ultimately every quantity of physics, including space and time, will turn out to be discrete and finite." Fredkin, together with Steven Wolfram, are strong promoters of a CA-based physics.In 2016 Gerard 't Hooft published an idea of rebuilding quantum mechanics using CA (Hooft, 2016; Berto, Rossi, & Tagliabue, 2010).

2.1 Elementary cellular automata

As mentioned earlier, elementary CA are one-dimensional binary CA with neighborhood of size one and thay can be labeled as follows. We have the local rule

$$f(p, q, r) = \beta, \tag{1}$$

where $p, q, r, \beta \in \{0, 1\}$, so we could get eight different configurations in the neighborhood $\mathcal{U}_1(i) = \{i-1, i, i+1\}$, like this:

$$(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), ..., (1, 1, 1). \tag{2}$$

If $\beta_0, \beta_1, ..., \beta_7 \in \{0, 1\}$ are the corresponding values of β, then the cellular automaton with the local rule f can be conclusively identified by the number

$$ID = \sum_{i=0}^{7} \beta_i 2^i \in \{0, 1, ..., 255\}. \tag{3}$$

This means, that there are 256 different elementary CA. Other way to calculate all possible configuration could be following. To define a local rule, we have to specify the update state of the central cell given all possible configuration of its local local neighborhood. Since there are eight such configurations and two states, the amount of all possible assignments is $2^8 = 256$. For example, the cellular automaton with local rule

$$\begin{aligned} f(0,0,0) &= 0, & f(1,0,0) &= 0,\\ f(0,0,1) &= 1, & f(1,0,1) &= 1,\\ f(0,1,0) &= 1, & f(1,1,0) &= 1,\\ f(0,1,1) &= 1, & f(1,1,1) &= 0, \end{aligned} \tag{4}$$

is coded as the decimal number

$$\begin{aligned} ID = 0 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 1 \times 2^3 + 0 \times 2^4\\ +1 \times 2^5 + 1 \times 2^6 + 0 \times 2^7 = 110. \end{aligned} \tag{5}$$

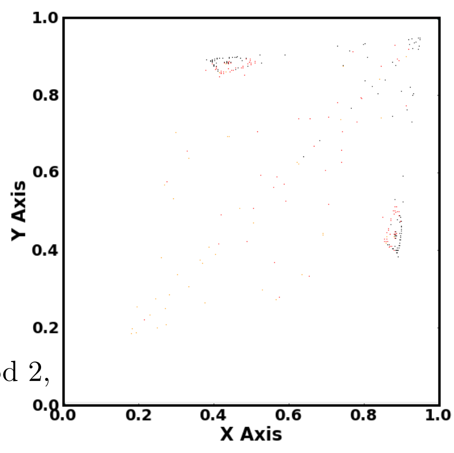
We could get the local rule $f(p, q, r) = \beta$ of an elementary cellular automaton from its identification number recursively:

$$\beta_0 = \text{ID mod } 2,$$

$$\beta_i = \frac{\text{ID} - \beta_0 - ... - \beta_{i-1}2^{i-1}}{2^i} \text{mod } 2,$$

$1 \leq i \leq 7.$

Figure 4: The ninth iteration of the CML model (13) as CA for $\mu = 3.8$ and $\epsilon = 0.09$



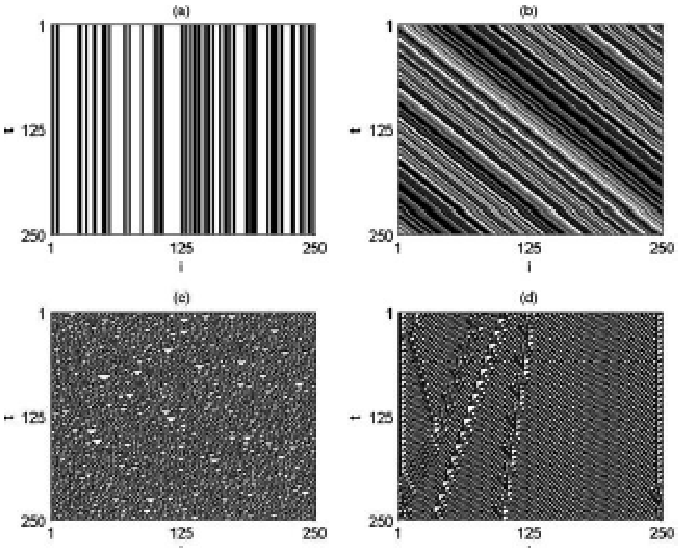


Figure 5: Elementary CA

So, for determining the evolution of these CA, we need only eight bits β_i, no closed formula for f is necessary (Chua, Sbitnev, & Yoon, 2003). Stephen Wolfram studied very deeply behavior of all 256 elementary CA. He calculated the time evolution with each local rule and initial configuration, until it settled at stable pattern of behaviour (Wolfram, 1984, 2002). Wolfram sorted elementary CA in four classes (see in Figure 5) by increasing complexity:

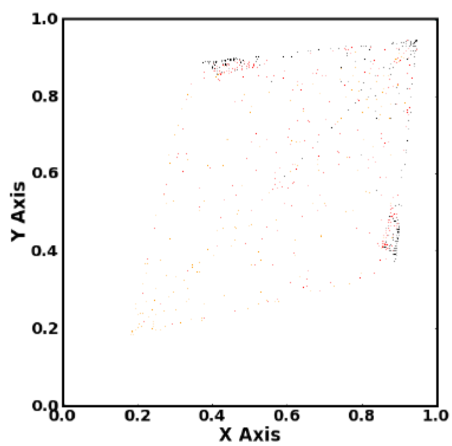
1. The configurations converge to fixed point (a)
2. Time evolution yields a sequence of simple stable or periodic structures (b)
3. The behavior is chaotic (c)
4. Time evolution yields localized structures that move around and interact witheach other in very complicated ways (d)

2.2 Topological Entropy of CA

Topological entropy is measure of complexity of F and the spatiotemporal complexity of a cellular automaton can be measured by that:

$$\mathrm{h}_{top}(F) = \lim_{w\to\infty} \lim_{x\to\infty} \frac{1}{t} \log R(w,t), \tag{6}$$

Figure 6: The eighth iteration of the CML model (13) as CA for $\mu = 3.8$ and $\epsilon = 0.09$



where $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ and $R(w, t)$ is the number of distinct rectangles of width w and height (temporal extent) t occurring in a space–time evolution diagram of $(S^{\mathbb{Z}}, F)$ and S a finite alphabet from which automaton is taking on values.

This is way, how to calculate $\mathrm{h}_{top}(F)$ by means of the topological entropy of a two-dimensional interval map. Set $\Omega = S^{\mathbb{Z}}$, where $S = \{0, 1, ..., |S| - 1\}$ in the case of a one-dimensional cellular automaton with $|S|$ states, and define similar to the map $\phi_{|S|} = (\phi^-_{|S|}, \phi^+_{|S|}) : S^{\mathbb{Z}} \to [0, 1]^2$,

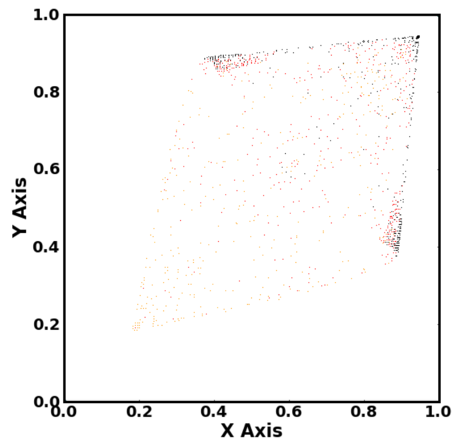
$$\phi_{|S|} : x_t \mapsto (\phi^-_{|S|}(x^-_t), \phi^+_{|S|}(x^+_t)), \tag{7}$$

where $x_t = (x_t(i))_{n\in\mathbb{Z}}$, $(x_t(-i))_{i\in\mathbb{N}}$ is the left sequence of x_t, $x^+_t = (x_t(i))_{i\in\mathbb{N}_0}$ is the corresponding right sequence, the component maps $\phi^-_{|S|} : S^{\mathbb{N}} \to [0, 1]$, $\phi^+_{|S|} : S^{\mathbb{N}_0} \to [0, 1]$ are given by

$$\phi^-_{|S|}(x^-_t) = \sum_{i=1}^{\infty} \frac{x_t(-i)}{|S|^i}, \phi^+_{|S|}(x^+_t) = \sum_{i=0}^{\infty} \frac{x_t(i)}{|S|^{i+1}} \tag{8}$$

and the bisequences $x_t = (x^-_t, x^+_t)$ are lexicographically ordered. The map $\phi_{|S|}$ is an order isomorphism ($[0, 1]^2$ being lexicographically ordered), up to a measure zero set $\mathcal{N}$ which comprises those bisequences whose left or right sequences terminate in $1, 0^{\infty}$ or $0, (|S|-1)^{\infty}$. Also, it is quite easy to check that $\phi_|S|$ is a homeomorphism from $S^{\mathbb{Z}}\backslash\mathcal{N}$ to its range.

Figure 7: The seventh iteration of the CML model (13) as CA for $\mu = 3.8$ and $\epsilon = 0.09$



3 Coupled map lattices

A coupled map lattice (CML) is a dynamical system that models the behavior of non-linear systems (especially partial differential equations). They are mainly used to study the chaotic dynamics of spatially extended systems. This includes the dynamics of spatiotemporal chaos where the number of effective degrees of freedom diverges as the size of the system increases (Chazottes & Fernandez, 2005). CML is a dynamical system with discrete time, space a continuous states. Interactions of cells could be done by for example electric charge, vibrations, magnetism or others. This can be done in several different ways so, we don't need wires between elements. This is quite distant from processors used in todays computers. Difference between CML and CA is also, that state of cells in CA depends on their neighbours, but CML is not neighbor dependent.

In 1983 the following model was introduced by Kaneko (Chazottes & Fernandez, 2005):

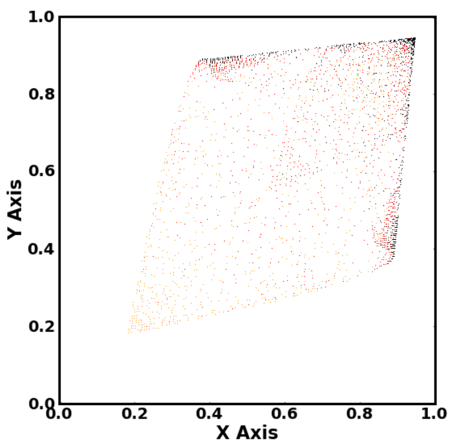
$$u_s^{t+1} = (1-\epsilon)f(u_s^t) + \frac{\epsilon}{2}(f(u_{s+1}^t) + f(u_{s-1}^t)), t \in \mathbb{N}, \epsilon \in [0,1] \tag{9}$$

where $u_s^t \in \mathbb{R}$ and f is a real mapping. Kaneko used this model to study spaciotemporal complexity (turbulance, convection, etc.).

The mapping f was chosen according to the local behaviour describing the studied phenomenom.

Mapping is a recursive function of two terms, an individual non-linear reaction, and a spatial interaction of variable intensity. A lot of wok in CML is based in weak coupled systems, where are studied diffeomorphisms of the state space, which are close to identity. Weak coupling with monotonic dynamical regimes represents spatial chaos phenomena and they are also quite popular in neural models. Weak coupling unimodal maps could be shown by their stable periodic points and they found use in gene regulatory network models (Wiener, 1946). Space-time chaotic phenomena could be shown from chaotic mappings subject to weak coupling coefficients and their popularity is high in phase transition phenomena models (Bunimovich & Sinai, 1988).

Figure 8: The sixth iteration of the CML model (13) as CA for $\mu = 3.8$ and $\epsilon = 0.09$



Intermediate and strong coupling interactions are not that popular, because they don't have much results. With respect to fronts and traveling waves, riddled basins, riddled bifurcations, clusters and non-unique phases we study intermediate interactions. For modeling synchronization effects of dynamic spatial systems we use strong coupling interactions. We cannot catch the global or local coupling nature of the interaction with these classifications. They cannot also consider the frequency of the coupling which can exist as a degree of freedom in the system. And they cannot as well distinguish between sizes of the underlying space or boundary conditions (Courbage & Kamiński, 2015; Nozawa, 1992; Ho, Hung, & Jiang, 2004).

The principle of CML is in the reductionism in procedure, not in each elementary process. Physicists often try to adopt the reductionism in its exact sense, and they start from a microscopic level. We are not going to take such a simple viewpoint. Starting from a suitably rough description, we are introducing the reductionism in a macroscopic level. This procedural reductionism has been regard not in physics, but in biology and in artificial intelligence, as a functional module (Letichevskii & Reshod'ko, 1972; Hedlund, 1969). As far as fyzics is concerned, lets give an example. Assume that we have a phenomenon in fluids, which is created by a local chaotic process and diffusion. A simple reductionist could start from a microscopic model, like molecular dynamics or lattice gas cellular automata. Some others may trust only in equations in a coarse-grained level like Navier-Stokes equation. In CML we take a different look from both of the two; reductionism in procedure. We try to reduce the phenomena into local chaos and diffusion processes. Then we select a suitable lattice model in a semi-coarse grained level for each process. As the simplest choice we can adopt a local logistic map for chaos, and a discrete Laplacian operator for the diffusion. The former process is given by

$$x_n^{'}(i) = f(x_n(i)) \tag{10}$$

where $x_n(i)$ is a variable at time n and lattice site i, and $x_n^{'}(i)$ is introduced as the intermediate value. For the logistic map $f(x)$ is chosen to be $1 - ax^2$. The discrete Laplacian operator for diffusion is given by

$$x_{n+1}(i) = (1-\epsilon)x_n^{'}(i) + \frac{\epsilon}{2}(x_n^{'}(i+1) + x_n^{1}(i-1)) \tag{11}$$

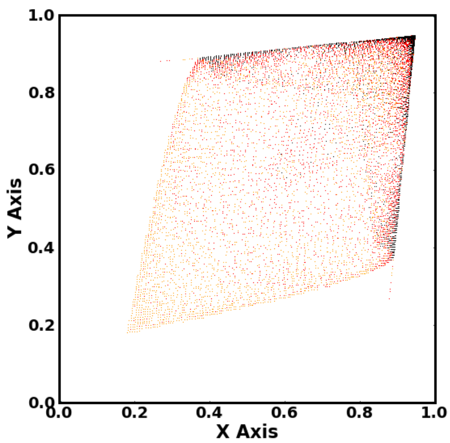
Combining the above two processes, our dynamics is given by

$$x_{n+1}(i) = \Big(1 - \epsilon f(x_n(i)\Big) + \frac{\epsilon}{2}\Big(f(x_n(i+1)) + f(x_n(i-1))\Big)$$

The above CML has been investigated a lot as a standard model for spatiotemporal chaos. We notice that the local chaos and diffusion processes are carried out separately in the above, which is one of the virtue of our CML (Kaneko, 1991).

The configurations u_s^t could be a part of velocity field or local density of population (Amigó, 2010). So those ex-

Figure 9: The fifth iteration of the CML model (13) as CA for $\mu = 3.8$ and $\epsilon = 0.09$



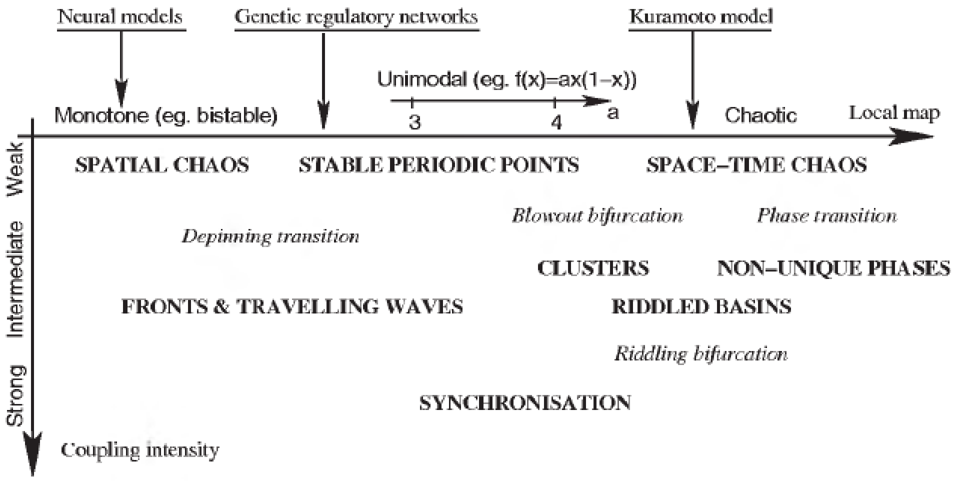


Figure 10: Schematic representation of the phenomenology of CML versus the local map and the coupling intensity(Chazottes & Fernandez, 2005)

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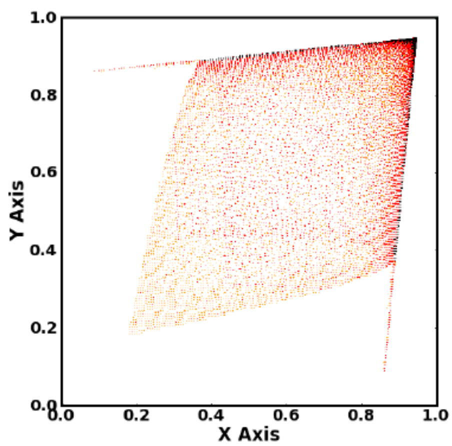
amples are bounded sequences, periodic or finite. Sometimes we however require unbounded configurations. In basic model we can see that CML is determined by two terms, f is a nonlinear reaction and ϵ is spatial interaction. One dimensional CML has six phases:

1. frozen random patterns,
2. pattern selection and suppression of chaos,
3. Brownian motion of defects,
4. defect turbulence,
5. pattern competition intermittency,
6. fully developed turbulence.

Figure 12: The fourth iteration of the CML model (13) as CA for $\mu = 3.8$ and $\epsilon = 0.09$



3.1 The logistic family

For the further ideas development some elementary notions from the theory of discrete dynamical system will be given. By a discrete dynamical system an ordered pair

$$(X, f)$$

is meant (Devaney & Eckmann, 1987). Here X stands for the state space, classically a compact metric space and f is

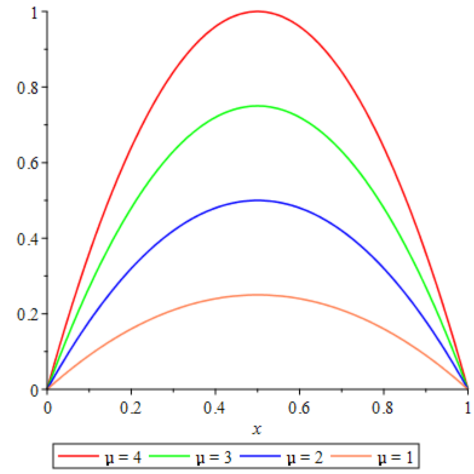


Figure 11: Graphs of the logistic map f_μ for $\mu = 1$, $\mu = 2$, $\mu = 3$ and $\mu = 4$.

a continuous map.

$$f : X \to X$$

that is into not necessarily auto. The main aim of the theory of discrete dynamical system focuses on behaviour of trajectories, that are received by iterations of given f. The n-th iteration of x under f is defined as

$$\underbrace{f^n(x) = f \circ f ... \circ f(x)}_{\text{n-times}}$$

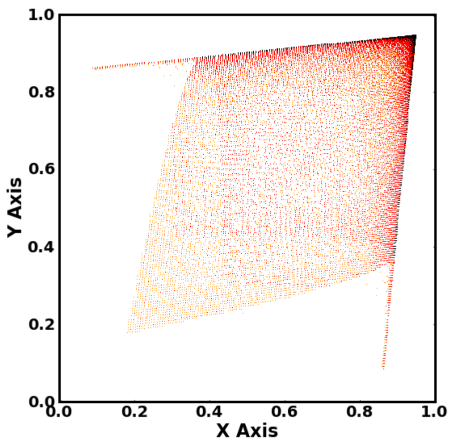
n-fold can position of f. Here f^0 stands for identity map on x. The crutial generic example, that is used in this thesis, is well known logistic family.

The family of logistic maps $f_\mu : [0,1] \to [0,1]$ where $0 \le \mu \le 4$, could be defined by:

$$f_\mu(x) = \mu x(1 - x). \tag{12}$$

The interval $I_\mu = [f_\mu^2(1/2), f_\mu(1/2)]$ is called *the core of* $f\mu$, when $\mu \in (2,4)]$. When the parameter $\mu \in [0,2]$, the interval I_μ does not have that nice properties, but the definition is still good (Lampart & Oprocha, 2016).For if $\mu = 0$ or $\mu = 2$ the core $I_0 = \{0\}, I_2 = \{\ 1/2\}$, respectively, degenerates into single point. The core I_μ is strongly invariant, that is $f_\mu(I_\mu) = I_\mu$, and every point from $(0,1)$ is attracted to I_μ. The dynamics on the core can be very rich (Brucks, Diamond, Otero-Espinar, & Tresser, 1991). For the family of tent maps the dynamics on the core is topologically exact for some range of parameters, which, in general, means that

Figure 13: The third iteration of the CML model (13) as CA for $\mu = 3.8$ and $\epsilon = 0.09$



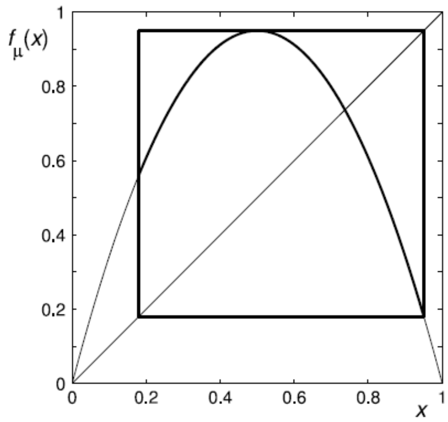


Figure 15: Graph of f_μ for $\mu = 3.8$ and the graph restricted to the core I_μ (bounded by box)

most rich dynamical behavior is present in the core. In those logistic maps, the calculations are not that simple and spectrum of possible dynamical behaviors is richer. However, it is known that for some parameters the dynamics on the core of the logistic map is the same (in the sense of topological conjugacy) as on the core of tent map with slope corresponding to μ.

3.2 CML system of Laplacian type

By (with coupling constant ϵ) the following map of the square $F : [0,1]^2 \to [0,1]^2$ is meant,

$$F(x, y) = (F_x(x, y), F_y(x, y)) \tag{13}$$

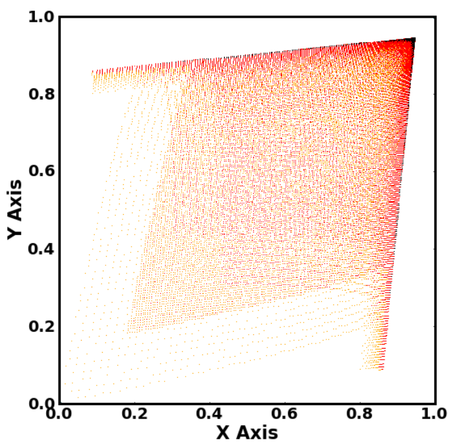
where

$$F_x(x, y) = (1-\epsilon)f_\mu(x) + \epsilon f_\mu(y), \tag{14}$$

$$F_y(x, y) = (1-\epsilon)f_\mu(y) + \epsilon f_\mu(x), \tag{15}$$

Simple calculations yield that $\mathrm{Fix}(F) \supseteq \{(0,0), (p_\mu, p_\mu)\}$ for $\mu > 1$, where $p_\mu = (\mu - 1)/\mu$ and they are the only fixed points of F on the diagonal. Note that the point p_μ is fixed for f_μ and is attracting or repulsive for $\mu \in (1,3)$ or $\mu \in (3,4)$, respectively. It may happen that $\mathrm{Fix}(F)$ contains also points outside diagonal, however always $\#\,\mathrm{Fix}(F) \le 4$ (Lampart & Oprocha, 2016; Kendall & Fox, 1998). Here, as usual, $\mathrm{Fix}(F)$ and $\mathrm{Per}(F)$ stand for the set of all fixed and periodic points of F , respectively; and the space $\mathbb{R}^2$ is endowed with the Euclidean norm $||.||$.

Figure 14: The second iteration of the CML model (13) as CA for $\mu = 3.8$ and $\epsilon = 0.09$



Let consider the set

$$R = (\mu, \epsilon) \in [0, 4] \times [0, 1] : 1 > \mu|1 - 2\epsilon|.$$

then

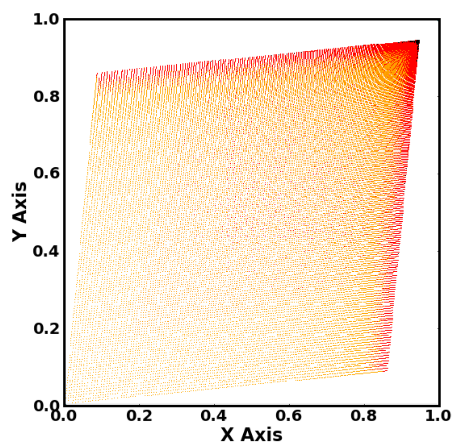
Proposition 1 *(Lampart & Oprocha, 2016) Denote by $\hat{\mu}$ the Feigenbaum constant. For any $x, y \in [0, 1]$ and $(\mu, \epsilon) \in R$ we have:*
(1) if $\mu \geq \hat{\mu}$ then
$\lim_{n\to\infty} \mathrm{dist}(F^n(x, y), \Delta(I_\mu) \cup \{(0, 0)\}) = 0,$
(2) if $\mu \in (2, \hat{\mu})$ and $(x, y) \in [0, 1]^2$ then either $(x, y) \in \{0, 1\}^2$ or there is $z \in I_\mu \cap \mathrm{Per}(f_\mu)$ such that $\lim_{n\to\infty} ||F^n(x, y) - F^n(z, z)|| = 0$.
(3) if $\mu \in (1, 2]$ and $(x, y) \in [0, 1]^2$ then either $(x, y) \in 0, 1^2$ or $\lim_{n\to\infty} F^n(x, y) = (p_\mu, p_\mu)$.
(4) if $\mu \leq 1$ then $\lim_{n\to\infty} F^n(x, y) = (0, 0)$.

It is clear that $\Delta = \{(x, x) : x \in [0, 1]\}$ is an invariant subset for F and that $F|_\Delta$ can be identified with f_μ by natural homeomorphism $\pi : [0, 1] \ni x \to (x, x) \in \Delta$, that is $F|_\Delta \circ \pi = \pi \circ f_\mu$ or equivalently $F|_\mu = \pi \circ f_\mu \circ \pi - 1$. It follows that, F is chaotic when $\mu = 4$ and many other values of $\mu \in [\hat{\mu}, 4)$, where $\hat{\mu} \approx 3.56994...$ is the Feigenbaum constant (smallest μ where f_μ has point of period $2n$ for every $n \in N$).

3.3 Coupled map lattices as Cellular automata

The main of the thesis is interpretationof the CML as CA with more than two states. Because state in CML could be any real number, we have to round to nearest number which computer could use. Main problem is rounding by computer. Our CA is implemented in Java language. Rounding causes small discrepancies. CML is implemented in grid of cells 500x500 and each cell contains small amount of points. In every step we calculate where will the cell transfer one of its points. Because the result of equation is between 0 and 1, we have to multiply the result by width and height of our grid, which is 500 in our example.

Figure 16: The first iteration of the CML model (13) as CA for $\mu = 3.8$ and $\epsilon = 0.09$



4 Implementation in Java

This section, as a main result part, is devoted to the implementation of the CML of the Laplacian type as CA in Java. The first part is import of packages, see in Listing 1. Next is part, where colours are created. According to colour we could determine the amount of point in one cell, see in Listing 2. In third part are defined some variables for calculations or arrays which represents cells, see in Listing 3. In another part we create the space for our CML. We fulfill the cells with points give them a colour according the amount, see in Listing 4. The last is the main part, where we iterate our model. Here is defined behaviour of our model, see in Listing 5.

Whole model is implemented in Java language, which is object-oriented language. It is very easy to learn and easy to understand and can be used for solving variety of problems. This language is very robust, safe and universal. On the other hand, Java language was not designed to solve mathematical problems, so it turned out, that is not best choice for this problem. Implementation could be divided into two main parts. In first part environment and graphic user interface is created. It is quite simple and straightforward. Second part, implementation, is in the last listing. Very important is to have more than one point per cell in the beginning, because without them the model would not depict completely.

```java
package cml;

import java.awt.BorderLayout;
import java.awt.Color;
import java.awt.Dimension;
import java.awt.Graphics;
import java.awt.event.ActionEvent;
import java.awt.event.ActionListener;
import java.util.ConcurrentModificationException;

import javax.swing.JButton;
import javax.swing.JFrame;
import static javax.swing.JFrame.EXIT_ON_CLOSE;
import javax.swing.JLabel;
import javax.swing.JPanel;
import javax.swing.Timer;
```

Listing 1: Import of packages of the CML model in Java.

Here are imported packages.

```java
public class CML extends JFrame{
```

```java
private static final Color white = Color.WHITE, black = Color.BLACK, red =
    Color.RED;
private final Color[] colors;

 private final Board board;
 private final JButton start_pause;
 JLabel label_it;
 int it = 1;     //Iteration counter

 public CML(){
     // Colors for scaling
     colors = new Color[10];
     colors[9] = Color.decode("0xff1100");
     colors[8] = Color.decode("0xff2200");
     colors[7] = Color.decode("0xff3300");
     colors[6] = Color.decode("0xff4400");
     colors[5] = Color.decode("0xff5500");
     colors[4] = Color.decode("0xff6600");
     colors[3] = Color.decode("0xff7700");
     colors[2] = Color.decode("0xff8800");
     colors[1] = Color.decode("0xff9900");
     colors[0] = Color.decode("0xffaa00");
     board = new Board();
     board.setBackground(white);

     label_it = new JLabel("Iteration: "+it);

     start_pause = new JButton("Start");
     start_pause.addActionListener(board);

     this.add(board, BorderLayout.NORTH);
     this.add(start_pause, BorderLayout.SOUTH);
     this.add(label_it, BorderLayout.CENTER);
     this.setDefaultCloseOperation(EXIT_ON_CLOSE);
     this.pack();
     this.setVisible(true);
```

```
}
```

Listing 2: Creating colours and labels of the CML model in Java.

Second part is where colours are created. Acording to colour we could determine the amount of point in one cell.

```
private class Board extends JPanel implements ActionListener{

    private final Dimension DEFAULT_SIZE = new Dimension(500, 500); //board
         is 500x500 pixels
    private final int DEFAULT_CELL = 1, DEFAULT_TIME = 1;         //size of
        cell in pixels, time is in miliseconds
    private final double epsilon = 0.06;
    private final double mi = 3.8;

    private final Dimension board_size;
    private final int cell_size, time;
    private boolean run;
    private final Timer timer;

    private final Color[][] grid; //this contain a color of cell
    private final int[][] state; //this contain number of points in cell
```

Listing 3: Defining variables of the CML model in Java.

In this part are defined some variables for calculations or arrays which represents cells.

```
    public Board(){
        board_size = DEFAULT_SIZE;
        cell_size = DEFAULT_CELL;
        time = DEFAULT_TIME;
        run = false;


        grid = new Color[board_size.height + 1][board_size.width + 1];
        state = new int[board_size.height + 1][board_size.width + 1];
        for (int h = 0; h < board_size.height; h++)
            for (int w = 0; w < board_size.width; w++){
                    grid[h][w] = colors[8]; //starting color for 2 points
                    state[h][w] = 2;      //each cell start with two points
```

```
        }

    timer = new Timer(time, this);
}

@Override
public Dimension getPreferredSize(){
    return new Dimension(board_size.height * cell_size, board_size.width
         * cell_size);
}

@Override
public void paintComponent(Graphics g){

    super.paintComponent(g);
    for (int h = 0; h < board_size.height; h++)
        for (int w = 0; w < board_size.width; w++){
            try{
                g.setColor(grid[h][w]);
                g.fillRect(h * cell_size, w * cell_size, cell_size,
                    cell_size);
            } catch (ConcurrentModificationException cme){}
        }
}
```

Listing 4: Creating board of the CML model in Java.

In this part is created the space for CML. Cells are fullfiled with points and colour is given to them according the amount.

```
@Override
public void actionPerformed(ActionEvent e) {

    if (e.getSource().equals(timer)){
        repaint();

        label_it.setText("Iteration: "+it);
        double x;
        double y;
```

```
for (int h = board_size.height; h >= 0 ; h--){
    for (int w = board_size.width; w >= 0 ; w--){

        //converting position in array into position in plane
        double x1 = h/(double)board_size.height;
        double y1 = w/(double)board_size.width;

        //calculating equation, the constant 500 is there because
             of rotation
        x = (     (1-epsilon)*mi*x1*(1-x1)+epsilon*mi*y1*(1-y1) )*
            board_size.height;
        y = 500-(     (1-epsilon)*mi*y1*(1-y1)+epsilon*mi*x1*(1-x1
            ) )*board_size.width;

        //cutting of the real part, storing whole number
        int xInt = (int)x;
        int yInt = (int)y;

        //value for comparing for rounding
        double a = xInt + 0.5;
        double b = yInt + 0.5;

        //rounding itself
        if (x >= a)
        {
            xInt++;
        }
        if (y >= b)
        {
            yInt++;
        }

        //moving points
        if(state[h][w] > 0)
        {
            state[xInt][yInt] ++;
            state[h][w]--;
        }
```

```
                    if ( state[xInt][yInt] > 9)
                    {
                        grid[xInt][yInt] = red;  //if there is high
                            concentration of points
                        if ( state[xInt][yInt] > 100) //if there is very high
                             concentration of points
                            grid[xInt][yInt] = black;
                    }
                    else if(state[xInt][yInt]==0)
                    {
                        grid[xInt][yInt] = white;
                    }
                    else
                    grid[xInt][yInt] = colors[(state[xInt][yInt])/8]; //one
                        scale for each eight points

                    grid[h][w] = white;

                }
            }

            it++;
        }

        else if(e.getSource().equals(start_pause)){
            if(run){
                timer.stop();
                start_pause.setText("Start");
            }
            else {
                timer.restart();
                start_pause.setText("Pause");
            }
            run = !run;

        }
    }
  }
}
```

Listing 5: Iteration of the CML model in Java.

This is the main part, where the model is iterated. Here is defined behaviour of this model.

5 Conclusions

In this thesis, cellular automata and coupled map lattices were studied. Cellular automaton as mathematical visualization instrument of the evolution of discrete dynamical systems was used. The theory of cellular automata was deeply studied and than applied to the coupled map lattice model of the Laplacian type. The main part of thesis implementation of this cellular automaton was done in Java language for the CML model (13) and variables $\mu = 3.8$ and $\epsilon = 0.09$.

In Section 2 it was given a general framework of cellular automaton. Furthermore, it was introduced elementary cellular automata and the theory of general cellular automata. Section 3 focused on theory of a coupled map lattices. The logistic family and also CML system of the Laplacian type was briefly introduced. The evolution of CML system generated by CA was performed. In Section 4 implementation of CML as CA was done, so we achieved our goal and manage to use cellular automaton as visualization instrument of the evolution of dynamical systems, thanks to the algorithm.

References

Amigó, J. (2010). *Permutation complexity in dynamical systems: ordinal patterns, permutation entropy and all that.* Springer Science & Business Media.

Berto, F., Rossi, G., & Tagliabue, J. (2010). *The mathematics of the models of reference.* College Publications.

Brucks, K., Diamond, B., Otero-Espinar, M., & Tresser, C. (1991). Dense orbits of critical points for the tent map. *Contemp. Math*, *117*, 57–61.

Bunimovich, L., & Sinai, Y. G. (1988). Spacetime chaos in coupled map lattices. *Nonlinearity*, *1*(4), 491.

Chapman, P. (2002). Life universal computer. *http://www. igblan. free-online. co. uk/igblan/ca.*

Chazottes, J.-R., & Fernandez, B. (2005). The cml2004 project. In *Dynamics of coupled map lattices and of related spatially extended systems* (pp. 1–8). Springer.

Chowdhury, D. R., Basu, S., Gupta, I. S., & Chaudhuri, P. P. (1994). Design of caecc-cellular automata based error correcting code. *IEEE Transactions on Computers*, *43*(6), 759–764.

Chua, L. O., Sbitnev, V. I., & Yoon, S. (2003). A nonlinear dynamics perspective of wolfram's new kind of science part ii: Universal neuron. *International Journal of Bifurcation and Chaos*, *13*(09), 2377–2491.

Coombes, S. (2009). The geometry and pigmentation of seashells. *Techn. Ber. Department of Mathematical Sciences.*

Courbage, M., & Kamiński, B. (2015). Directional metric entropy and lyapunov exponents for dynamical systems generated by cellular automata. In *Nonlinear dynamics new directions* (pp. 11–27). Springer.

Crutchfield, J. P. (1994). The calculi of emergence: computation, dynamics and induction. *Physica D: Nonlinear Phenomena*, *75*(1-3), 11–54.

Devaney, R. L., & Eckmann, J.-P. (1987). An introduction to chaotic dynamical systems. *Physics Today*, *40*, 72.

Gardner, M. (1970). Mathematical games. *Scientific American*, *222*(6), 132–140.

Goles, E., & Martínez, S. (2013). *Cellular automata, dynamical systems and neural networks* (Vol. 282). Springer Science & Business Media.

Hedlund, G. A. (1969). Endomorphisms and automorphisms of the shift dynamical system. *Mathematical systems theory*, *3*(4), 320–375.

Ho, M.-C., Hung, Y.-C., & Jiang, I.-M. (2004). Phase synchronization in inhomogeneous globally coupled map lattices. *Physics Letters A*, *324*(5-6), 450–457.

Hooft, G. (2016). *The cellular automaton interpretation of quantum mechanics* (Vol. 185). Springer.

Ilachinski, A. (2001). *Cellular automata: a discrete universe.* World Scientific Publishing Company.

Kaneko, K. (1991). Coupled map lattice. In R. Artuso, P. Cvitanović, & G. Casati (Eds.), *Chaos, order, and patterns* (pp. 237–247). Boston, MA: Springer US. doi: 10.1007/978-1-4757-0172-2_10

Kemeny, J. G. (1955). Man viewed as a machine. *Scientific American*, *192*(4), 58–67.

Kendall, B. E., & Fox, G. A. (1998). Spatial structure, environmental heterogeneity, and population dynamics: analysis of the coupled logistic map. *Theoretical population biology*, *54*(1), 11–37.

Kier, L. B., Seybold, P. G., & Cheng, C.-K. (2005). *Modeling chemical systems using cellular automata* (Vol. 1). Springer Science & Business Media.

Lampart, M., & Oprocha, P. (2016). Chaotic sub-dynamics in coupled logistic maps. *Physica D: Nonlinear Phenomena*, *335*, 45–53.

Letichevskii, A., & Reshod'ko, L. (1972). N. wiener's theory of the activity of excitable media. *Cybernetics*, *8*(5), 856–864.

Mitchell, M. (2002). Is the universe a universal computer? *Science*, *298*(5591), 65–69.

Muhtaroglu, A. (1996). *Cellular atomation processor based systems for genetic sequence comparison/database searching.* Cornell University, August.

Nozawa, H. (1992). A neural network model as a globally coupled map and applications based on chaos. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *2*(3), 377–386.

Pickover, C. A. (2009). *The math book: from pythagoras to the 57th dimension, 250 milestones in the history of mathematics.* Sterling Publishing Company, Inc.

Schiff, J. L. (2011). *Cellular automata: a discrete view of the world* (Vol. 45). John Wiley & Sons.

Toffoli, T., & Margolus, N. (1987). *Cellular automata machines: a new environment for modeling.* MIT press.

Tomassini, M., Sipper, M., & Perrenoud, M. (2000). On the generation of high-quality random numbers by two-dimensional cellular automata. *IEEE Transactions on computers*, *49*(10), 1146–1151.

Von Neumann, J. (1951). The general and logical theory of automata. *Cerebral mechanisms in behavior*, *1*(41), 1–2.

Von Neumann, J., & Burks, A. W. (1996). *Theory of self-reproducing automata.* University of Illinois Press Urbana.

Wiener, N. (1946). The mathematical formulation of the problem of conduction of impulses in a network of connected excitable elements, specifically in cardiac muscle. *Arch Inst Cardiol Mex*, *16*, 1–61.

Wolfram, S. (1984). Universality and complexity in cellular automata. *Physica D: Nonlinear Phenomena*, *10*(1-2), 1–35.

Wolfram, S. (2002). *A new kind of science* (Vol. 5). Wolfram media Champaign.